

Effective Coupling Constant in Renormalization Group for the Quantum Electrodynamics

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Abstract

Effective coupling constant in quantum electrodynamics is investigated. A pole appears in the effective coupling constant for the space-like momentum if it is calculated by perturbation. The pole can be eliminated by the analytic regularization. For QED the effective coupling constant is written in terms of the scale parameter, Λ , having the dimension of mass as in the case of QCD. Λ is determined by comparing with the experimental data. The calculated result agrees with experiment with $\Lambda \approx 1.64 \times 10^{47}$ GeV; it is very large but much smaller than the mass scale of Landau ghost. .

1 Introduction

The effective coupling constant, α_S , is one of the most fundamental quantities in the renormalization group. It is a renormalization group invariant quantity so that it is usually used to express the Green function. α_S is given in terms of the beta function, β , the derivative of the renormalized coupling constant with respect to the renormalization point. Actually, it is defined implicitly by the equation

$$\log \left(\frac{Q^2}{\mu^2} \right) = \int_{\alpha_r}^{\alpha_S(Q^2)} \frac{1}{\beta(\alpha')} d\alpha'. \quad (1.1)$$

Here, Q^2 is the squared momentum for the space-like region, μ is the renormalization point and the renormalized coupling constant α_r is given as $\alpha_S(\mu^2) = \alpha_r$. In this paper we restrict ourselves to the abelian and non-abelian gauge theories, the quantum electrodynamics (QED) and the quantum chromodynamics (QCD).

The beta functions are calculated by perturbation with recourse to the $\overline{\text{MS}}$ renormalization scheme. For QED, Gorishny et al. [1] performed calculation up to the fourth order approximation and obtained

$$\beta_{QED}(\alpha)/8\pi = \beta_0(\alpha/4\pi)^2 + \beta_1(\alpha/4\pi)^3 + \beta_2(\alpha/4\pi)^4 + \beta_3(\alpha/4\pi)^5 + \cdots, \quad (1.2)$$

where β_k , $k = 1, 2, \cdots, 4$ are

$$\begin{aligned} \beta_0 &= 2n_f/3\pi, \\ \beta_1 &= n_f/2\pi^2, \\ \beta_3 &= -(1 + 22n_f/9)n_f/16\pi^3, \\ \beta_4 &= -n_f \left[23 - (380/27 - 416\zeta(3))n_f/9 + 616n_f^2/243 \right], \end{aligned} \quad (1.3)$$

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with n_f being the number of flavor and $\zeta(n)$ the Riemann ζ function. For QCD, the gauge group being $SU(3)$ with the number of flavor $n_f = 3$, the three loop calculation was done by Tarasov et al. [2]. Their result is

$$\beta_{QCD}/4\pi = \beta_0(\alpha/4\pi)^3 + \beta_1(\alpha/4\pi)^5 + \beta_2(\alpha/4\pi)^7 + \dots, \quad (1.4)$$

where

$$\begin{aligned} \beta_0 &= (-11 + 2n_f/3), \\ \beta_1 &= -102 + 38n_f/3, \\ \beta_2 &= -2857/2 + 25033n_f/18 - 325n_f^2/54. \end{aligned} \quad (1.5)$$

One of the most important results in the renormalization group is that the qualitative feature of the effective coupling constant is determined by the sign of the lowest order term in the perturbation expansion, namely, the sign of β_0 . When $\beta_0 < 0$, the theory is asymptotically free and the $\alpha(Q^2) \rightarrow 0$ for $Q^2 \rightarrow \infty$. For $\beta_0 > 0$ the theory is asymptotically non-free. QCD with $n_f < 33/2$ is asymptotically free theory, but QED is not asymptotically free.

2 Analytic regularization

To simplify the problem, we start by studying the one loop approximation. Taking the lowest order term in the β function, we obtain the effective coupling constant: For QED, the integration of (1.1) leads to

$$\log(Q^2/\mu^2) = \frac{2\pi}{\beta_0} \left(\frac{1}{\alpha_r} - \frac{1}{\alpha_S(Q^2)} \right).$$

Writing the renormalized coupling constant α_r in terms of a parameter Λ , which is defined by the equation

$$\Lambda^2 = \mu^2 \exp \left(\frac{2\pi}{\beta_0 \alpha_r} \right), \quad (2.1)$$

we have

$$\alpha_S(Q^2)^{QED} = \frac{2\pi}{\beta_0} \frac{1}{\log(\Lambda^2/Q^2)}. \quad (2.2)$$

As $\beta_0 > 0$ for QED, the effective coupling constant $\alpha_S(Q^2)$ is an increasing function of Q^2 for $Q^2 < \Lambda^2$. $\alpha_S(Q^2)$ is singular at $Q^2 = \Lambda^2$ and becomes unphysical if $Q^2 > \Lambda^2$ because $\alpha_S(Q^2)$ turns out to be negative. For QCD, Q^2 dependence is different from QED as $\beta_0 (= -\beta'_0) < 0$. For the one loop approximation the scale parameter Λ is given as

$$\Lambda^2 = \mu^2 \exp \left(-\frac{4\pi}{\beta'_0 \alpha_r} \right) \quad (2.3)$$

and

$$\alpha_S(Q^2)^{QCD} = \frac{4\pi}{\beta'_0} \frac{1}{\log(Q^2/\Lambda^2)}. \quad (2.4)$$

It is a decreasing function of Q^2 and has a pole at $Q^2 = \Lambda^2$. It becomes negative and unphysical for $Q^2 < \Lambda^2$.

To eliminate the singularity of the effective coupling constant in gauge theories we have adopted the analytic regularization [3]. The regularization is performed in the following way: First calculate the effective coupling constant for the space-like region as a function of the squared momentum Q^2 , which is transformed to the time-like squared momentum t by the replacement

$$Q^2 \rightarrow e^{-i\pi} t. \quad (2.5)$$

For QCD the regularized effective coupling constant α_R is defined by the dispersion integral

$$\alpha_R^{QCD}(t) = \frac{1}{\pi} \int_0^\infty \frac{\sigma(t')}{t' - t} dt', \quad (2.6)$$

where

$$\sigma(t) = \frac{1}{2i}[\alpha_S(e^{-i\pi}t) - \alpha_S(e^{i\pi}t)]. \quad (2.7)$$

Therefore, the spectral function of QCD is given for the one loop approximation as

$$\sigma(t) = \frac{4\pi}{\beta'_0} \frac{1}{(\log(t/\Lambda^2)) + \pi^2}, \quad (2.8)$$

which is a decreasing function of t . Performing integration of (2.6) by means of the Cauchy theorem, we have

$$\alpha_R^{QCD}(Q^2) = \alpha_S^{QCD}(Q^2) - \frac{4\pi}{\beta_0} \frac{\Lambda^2}{Q^2 - \Lambda^2}. \quad (2.9)$$

For QED the spectral function σ turns out to be negative, and an increasing function of t

$$\sigma(t) = -\frac{2\pi}{\beta_0} \frac{1}{(\log(\Lambda^2/t))^2 + \pi^2}. \quad (2.10)$$

Consequently, the regularization should be changed from that of QCD. We take once subtracted dispersion formula

$$\alpha_R^{QED}(t) = \frac{t}{\pi} \int_0^\infty \frac{\sigma(t')}{t'(t' - t)} dt'. \quad (2.11)$$

We have the following regularized effective coupling constant for the one loop approximation:

$$\alpha_R^{QED} = \alpha_S^{QED}(Q^2) - \frac{2\pi}{\beta_0} \frac{Q^2}{\Lambda^2 - Q^2}. \quad (2.12)$$

Although we have considered one loop calculations so far, we may use (2.6) and (2.11) for general case.

For the case of QCD, the properties of effective coupling constant is investigated in [4]. So, we restrict ourselves to QED in this paper.

Here we use the effective coupling constant for the four loop calculation [1]. To make the formula simpler, we write the β function for QED

$$\beta(\alpha) = \alpha^2(a_1 + a_2\alpha + a_3\alpha^2 + a_4\alpha^3 + \dots). \quad (2.13)$$

Calculating (1.1), we obtain

$$\begin{aligned} -\frac{a_1}{2} \log\left(\frac{Q^2}{\mu^2}\right) &= \frac{1}{\alpha_S(Q^2)} - \frac{1}{\alpha_r} + \frac{a_2}{a_1} \log\left(\frac{\alpha_S(Q^2)}{\alpha_r}\right) - \frac{1}{a_1^2}(a_2^2 - a_1a_3)(\alpha_S(Q^2) - \alpha_r) \\ &+ \frac{1}{2a_3}(a_2^3 - 2a_1a_2a_3 + a_1^2a_4)(\alpha_S(Q^2) - \alpha_r^2) + \dots \end{aligned} \quad (2.14)$$

We eliminate the renormalized coupling constant α_r and the renormalization point μ by using the parameter with dimension of mass, Λ^{QED} , which is given by the equation

$$\frac{a_1}{2} \log\left[\frac{(\Lambda^{QED})^2}{\mu^2}\right] = \frac{1}{\alpha_r} - \frac{1}{a_1^2}(a_2^2 - a_1a_3)\alpha_r + \frac{1}{2a_1^3}(a_2^3 - 2a_1a_2a_3 + a_1^2a_4)\alpha_r^2 + \dots \quad (2.15)$$

We simply write Λ by omitting the superscript 'QED' hereafter. The effective coupling constant is now given as follows:

$$\begin{aligned} \frac{1}{\alpha_S(Q^2)} &= \frac{a_1}{2} \log(\Lambda^2/Q^2) + \frac{a_2}{a_1} \log\{\log(\Lambda^2/Q^2)\} \\ &+ \frac{2}{a_1^3 \log(\Lambda^2/Q^2)} [a_2^2 \log\{\log(\Lambda^2/Q^2)\} + a_2^2 - a_1a_3] \\ &- \frac{2}{a_1^5 \log^2(\Lambda^2/Q^2)} [a_2^3 \log^2\{\log(\Lambda^2/Q^2)\} \\ &- 2a_1a_2a_3 \log\{\log(\Lambda^2/Q^2)\} - a_2^3 + a_1^2a_4] + \dots \end{aligned} \quad (2.16)$$

3 Effective coupling constant of QED for the time-like momentum

We perform the analytic continuation of the effective coupling constant for the four loop calculation (2.16) by using the prescription given in the previous section. The squared momentum for the space-like region, Q^2 , is transformed to the time-like one t by the equation

$$Q^2 = e^{-i\pi}t.$$

(2.16) then becomes

$$1/\alpha_S(e^{-i\pi}t) = u + iv, \quad (3.1)$$

where u and v are given as follows:

$$\begin{aligned} u &= \frac{a_1}{2}R \cos \theta + \frac{a_2}{a_1} \log R + \frac{2}{a_1^3 R} [\cos \theta (a_2^2 \log R + a_2^2 - a_1 a_3) + \theta a_2^2 \sin \theta] \\ &\quad - \frac{2}{a_1^5 R} [\cos 2\theta \{a_2^3 (\log^2 R - \theta^2) - 2a_1 a_2 a_3 \log R - a_2^3 + a_1^2 a_4\} \\ &\quad + 2\theta \sin 2\theta (a_2^3 \log R - a_1 a_2 a_3)] \\ v &= \frac{a_1}{2}R \sin \theta + \theta \frac{a_2}{a_1} + \frac{2}{a_1^3 R} [-(a_2^2 \log R + a_2^2 - a_1 a_3) \sin \theta + \theta a_2^2 \cos \theta] \\ &\quad - \frac{2}{a_1^5 R} [-\sin 2\theta \{a_2^3 (\log^2 R - \theta^2) - 2a_1 a_2 a_3 \log R - a_2^3 + a_1^2 a_4\} \\ &\quad - \frac{2}{a_1^5 R} [-\sin 2\theta \{a_2^3 (\log^2 R - \theta^2) - 2a_1 a_2 a_3 \log R - a_2^3 + a_1^2 a_4\} \\ &\quad + 2\theta \cos 2\theta (a_2^3 \log R - a_1 a_2 a_3)], \end{aligned} \quad (3.2)$$

with

$$R = \sqrt{\log^2(\Lambda^2/t) + \pi^2} \quad (3.3)$$

and

$$\theta = \arctan(\pi / \log(\Lambda^2/t)) \quad (3.4)$$

Here the branch of $\arctan(\pi / \log(\Lambda^2/t))$ is taken to be continuous at $t = \Lambda^2$. $\text{Re } \alpha$ and $\text{Im } \alpha$ are given as

$$\text{Re } \alpha(t) = \frac{u}{u^2 + v^2}, \quad (3.5)$$

$$\text{Im } \alpha(t) = -\frac{v}{u^2 + v^2}. \quad (3.6)$$

We illustrate in Fig.1 (a), (b) $\text{Re } \alpha(t)$ and $\text{Im } \alpha(t)$ as functions of t for the time-like momentum, $t > 0$, respectively. Here, we use Λ which will be obtained by comparing with the experimental data in Sec.4. It must be remarked that the spectral function $\sigma(t)$ is given by $\sigma(t) = \text{Im } \alpha(t)$.

For the space-like momentum the effective coupling constant α_S has a pole at $Q^2 = Q^{*2}$ and the pole term is written as

$$\text{pole term of } \alpha_S = A^* \frac{Q^2}{Q^2 - Q^{*2}}. \quad (3.7)$$

Numerically, for the four loop calculation of (2.16) we have

$$A^* = 4.50088 \times 10^{-3}, \quad Q^{*2} = a^* \Lambda^2$$

with

$$a^* = 5.512275.$$

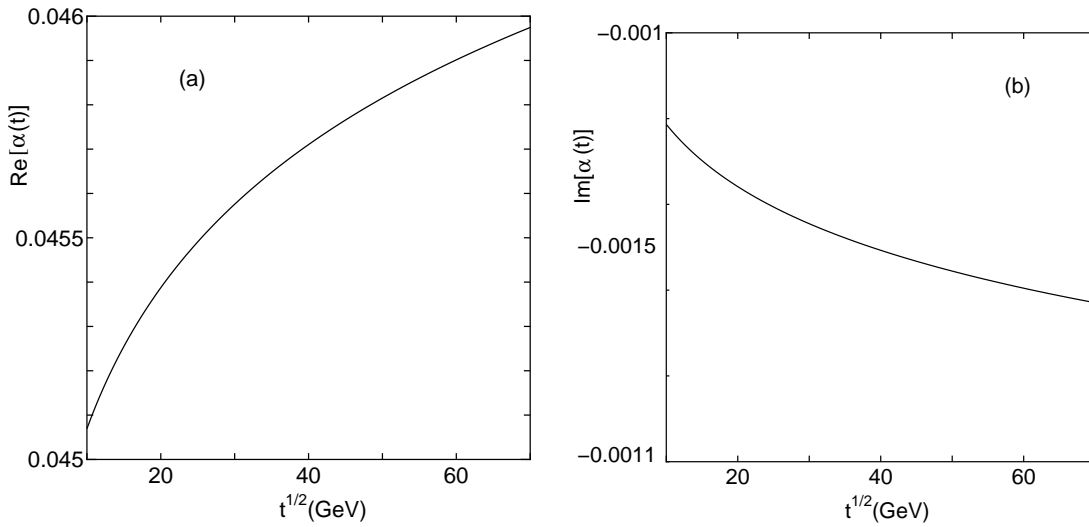


Fig.1 $\text{Re} \alpha$ and $\text{Im} \alpha$ for $\Lambda = 1.646 \times 10^{47}$ GeV for the time-like momentum. (a): $\text{Re} \alpha(t)$. (b): $\text{Im} \alpha(t)$.

The regularized effective coupling constant is obtained by subtracting the pole term (3.7), that is,

$$\alpha_R(Q^2) = \alpha_S(Q^2) - A^* \frac{Q^2}{Q^2 - Q^{*2}}. \quad (3.8)$$

Although the formula is exact only for the one loop approximation, it is approximately correct for the higher order calculations so long as $Q^2 \ll \Lambda^2$.

4 Comparison with experiments

We compare the effective coupling constant $\alpha_S(Q^2)$ with the experimental data [5] for the space-like and time-like momenta by taking Λ as an adjustable parameter. The experimental data imply that α_S is an increasing function of Q^2 . In Ref.[5] we have five data points for the space-like momentum and one for the time-like momentum. We use (2.16), for the former and $|\alpha_S(e^{-it})| = 1/\sqrt{u^2 + v^2}$, (3.1), for the latter and determine Λ so as to minimize χ^2 for the experimental data [5]. The parameter Λ is determined as follows:

$$\Lambda = 1.646 \times 10^{47} \text{ GeV}, \text{ with } \chi^2 = 4.22 \quad (\text{four loop approximation}),$$

where the degrees of freedom is 5.

We compare in Fig.2 (a) and (b) the calculated results with the experimental data of the effective coupling constant for QED for the space-like and time-like momentum, respectively.

Although we have used four loop calculation for the effective coupling constant, the experimental data are realized by the one loop approximation as well. The parameter Λ is then obtained to be

$$\Lambda = 2.166 \times 10^{49} \text{ GeV}, \text{ with } \chi^2 = 3.78. \quad (\text{one loop approximation}).$$

As the value of Q^2 is at most 10 GeV^2 for the existing experiments, the pole term in (3.8) can be neglected.

The value of Λ is much larger than that of QCD for which $\Lambda = O(1) \text{ GeV}$. It is, however, much smaller than the mass scale corresponding to the Landau ghost, where $\Lambda^{\text{Landau}} = m_e \exp(3\pi/2\alpha_r)$, with m_e being the electron mass and α_r the fine structure constant [6].

The large mass scale implies that some other particles than the electron is necessary such as Z and W or Higgs boson in the standard model.

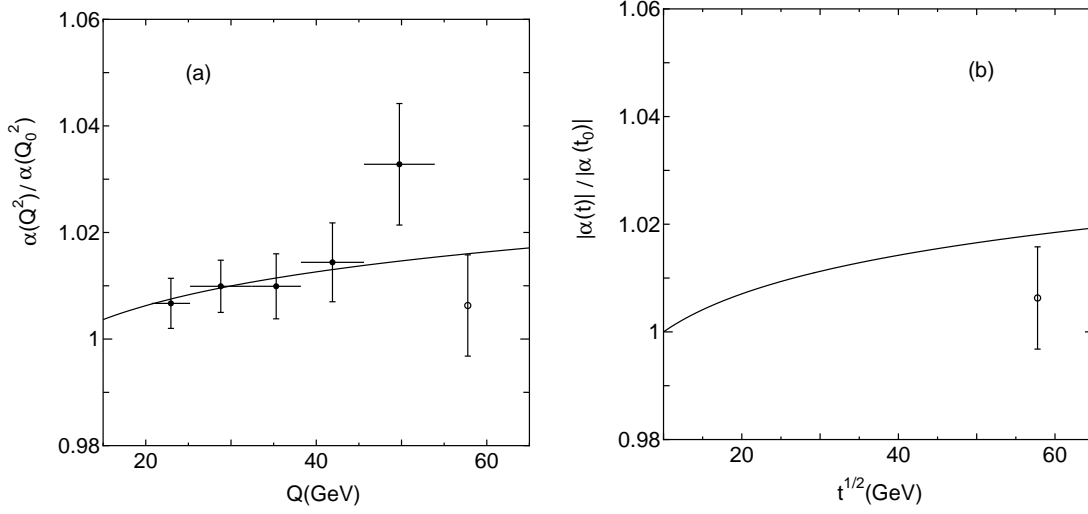


Fig.2 Effective coupling constant for QED with $\Lambda = 1.646 \times 10^{47}$ GeV. (a): $\alpha(Q^2)/\alpha(Q_0^2)$ for the space-like momentum with $Q_0 = 10$ GeV. The open circle is the data for the time-like momentum, the same point as in (b). (b): $|\alpha(t)|/|\alpha(t_0)|$ for the time-like momentum with $\sqrt{t_0} = 10$ GeV. Data points are taken from Re.[5]

To conclude the paper, we remark that the regularization of the effective coupling constant is similar to the Redmond and Uretsky regularization of the Green function in QED [7].

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